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A dynamical group $SU(2,2)$ and its use in the MIC–Kepler problem

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Abstract. It is widely known that the Kepler problem admits $SU(2,2)$ as a dynamical group. This article aims to show that $SU(2,2)$ is also a dynamical group for the MIC–Kepler problem, a generalization of the Kepler problem. It is already known that the symmetry groups for the MIC–Kepler problem are $SO(4)$, $SO_0(1,3)$, and $E(3)$, according to whether the energy is negative, positive, or zero. It is shown in this article that the double cover of the respective symmetry groups, $SU(2)\times SU(2)$, $SL(2,\mathbf{C})$, and $SU(2)\ltimes \mathbf{R}^3$, a semi-direct product, are realized as subgroups of $SU(2,2)$. Isoenergetic orbit spaces are also studied, which are defined to be quotient manifolds of the respective energy manifolds by the respective Hamiltonian flows. Each of the isoenergetic orbit spaces is shown to be realized as a (co-)adjoint orbit of the symmetry group. In addition, use of the isoenergetic orbit spaces is discussed. In fact, a certain class of perturbed MIC–Kepler problems are shown to induce dynamical systems on the isoenergetic orbit space. If the energy is negative, the generic isoenergetic orbit space is diffeomorphic with $S^2\times S^2$, so that the Euler number of $S^2\times S^2$ provides the number of singular points for the reduced perturbed system, and in turn that of closed orbits for the perturbed MIC–Kepler problem.

1. Introduction

A dynamical group $SU(2,2)$ has long been studied in relation to the Kepler problem, in particular on the level of Lie algebras. That is, the generators of the Lie algebra $su(2,2)\cong so(2,4)$ have been extensively discussed—see Györgyi (1968, 1969), Barut and Bornzin (1971), Tripathy *et al* (1975), Baumgarte (1978), Iosifescu and Scutaru (1980, 1984), for example. Study on the level of Lie groups has been made by Souriau (1974) and well exhibited in Guillemin and Sternberg (1977, 1990). Kummer (1982, 1983, 1985) employed the dynamical group $SU(2,2)$ in his series of papers on the perturbation of the Kepler problem. The present article aims to show that $SU(2,2)$ is also a dynamical group for the MIC–Kepler problem, an extension of the Kepler problem. The symmetry groups for the MIC–Kepler problem have been studied in a series of papers by Iwai and Uwano (1986, 1988, 1991, 1991) both in classical and quantum mechanics. According to their results, the MIC–Kepler problem has the same symmetry groups as the Kepler one; depending on whether the energy is negative, positive, or zero, the symmetry group is $SO(4)$, $SO_0(1,4)$, or $E(3)$, where $SO_0(1,3)$ is the identity component of the Lorentz group. This article shows that the double cover of the respective symmetry groups, i.e. $SU(2)\times SU(2)$, $SL(2,\mathbf{C})$, and $SU(2)\ltimes\mathbf{R}^3$, are realized as subgroups of $SU(2,2)$. In this sense, $SU(2,2)$

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can be interpreted as a dynamical group for the MIC–Kepler problem. To show these results, the reduction method is profoundly useful. This is because the MIC–Kepler problem is naturally defined as a reduced system.

The reduction method has another application in discussing isoenergetic orbit spaces. Indeed, by using the reduction procedure together with the momentum map associated with the symmetry group, all the isoenergetic orbit spaces are shown to be realized as (co-)adjoint orbits of the respective symmetry groups.

The isoenergetic orbit space finds good use in discussing a certain class of perturbed MIC–Kepler problems. In fact, a certain class of the perturbed MIC–Kepler problems gives rise to dynamical systems on the isoenergetic orbit space. In the case of negative energy, the topology of the isoenergetic orbit space provides information on the flow of the perturbed MIC–Kepler problem.

The organization of this paper is as follows:

Section 2 contains a review of the reduction of the phase space $T^*(\mathbf{R}^4 - \{0\})$, which is closely related with the Kustaanheimo–Stiefel (KS) transformation (Kustaanheimo and Stiefel 1965). By the use of a $U(1)$ symmetry, the standard phase space $(T^*(\mathbf{R}^4 - \{0\}), d\theta)$ is reduced to the phase space $(T^*(\mathbf{R}^3 - \{0\}), \sigma_\mu)$ with the symplectic form σ_μ other than the standard one.

Section 3 deals with the momentum map associated with $SU(2,2)$, which is a map of $\mathbf{C}^4$ to $su(2,2)^*$, the dual to the Lie algebra $su(2,2)$ of $SU(2,2)$.

In section 4, the reduced phase space $(T^*(\mathbf{R}^3 - \{0\}), \sigma_\mu)$ is shown to be symplectomorphic with a (co-)adjoint orbit of $SU(2,2)$, on which the Kirillov–Kostant–Souriau form is defined. Sections 2 to 4 are reviews of known results on the dynamical group $SU(2,2)$, which are, however, reformulated for the purpose of the study in following sections.

Section 5 gives the symmetry groups for the MIC–Kepler problem as subgroups of $SU(2,2)$, in each of the cases of negative, positive, and zero energies. To be strict, the double cover of each symmetry group is realized as a subgroup of $SU(2,2)$, which is $SU(2) \times SU(2)$, $SL(2,\mathbf{C})$, or $SU(2) \ltimes \mathbf{R}^3$, a semi-direct product, according to whether energy is negative, positive, or zero.

Section 6 deals with isoenergetic orbit spaces. Since all the orbits of the Hamiltonian flow on respective energy manifolds determine a group action, one can define an isoenergetic orbit space to be the quotient manifold by that group action. The reduction procedure and the momentum map associated with the respective symmetry groups are used to prove that the isoenergetic orbit space is symplectomorphic with a (co-)adjoint orbit of the symmetry group. In fact, according to whether the energy is negative, positive, or zero, the isoenergetic orbit space is realized as a (co-)adjoint orbit of $SU(2) \times SU(2)$, $SL(2,\mathbf{C})$, or $SU(2) \ltimes \mathbf{R}^3$, respectively.

Section 7 contains use of the isoenergetic orbit spaces. A certain class of perturbed MIC–Kepler problems are defined on the isoenergetic orbit spaces through the reduction procedure. If the energy is negative, the isoenergetic orbit space is diffeomorphic with $S^2 \times S^2$. The Euler number of $S^2 \times S^2$ gives the number of singular points of the Hamiltonian flow on $S^2 \times S^2$ and hence, in turn, provides the number of closed orbits for the perturbed MIC–Kepler problem of negative energy.

Section 8 contains concluding remarks on co-adjoint structure.

2. A review of the reduction of the phase space $T^*(\mathbf{R}^4 - \{0\})$

We consider the phase space $T^*(\mathbf{R}^4 - \{0\}) = (\mathbf{R}^4 - \{0\}) \times \mathbf{R}^4$ with the cartesian coordinates (x_j, y_j), $j = 1, \ldots, 4$. The standard symplectic form is defined to be

$$\mathrm{d}\theta = \sum_{j=1}^{4} \mathrm{d}y_j \wedge \mathrm{d}x_j \qquad \text{with} \qquad \theta = \sum_{j=1}^{4} y_j \mathrm{d}x_j . \tag{2.1}$$

A key to our reduction method is that the configuration space $\dot{\mathbf{R}}^4 := \mathbf{R}^4 - \{0\}$ is a $U(1)$-bundle:

$$U(1) \longrightarrow \dot{\mathbf{R}}^4 \stackrel{\pi}{\longrightarrow} \dot{\mathbf{R}}^3 \tag{2.2}$$

where $\dot{\mathbf{R}}^3 := \mathbf{R}^3 - \{0\}$. For $x = (x_j) \in \dot{\mathbf{R}}^4$, the action of $U(1) \cong SO(2)$ is defined to be

$$x \longmapsto T(t)x \tag{2.3}$$

where the matrix $T(t)$ is given by

$$T(t) = \begin{pmatrix} R(t) & 0 \\ 0 & R(t) \end{pmatrix} \qquad \text{with} \qquad R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} .$$

The projection π is realized by

$$q_1 = 2(x_1x_3 + x_2x_4) \qquad q_2 = 2(x_1x_4 - x_2x_3) \qquad q_3 = x_1^2 + x_2^2 - x_3^2 - x_4^2 \tag{2.4}$$

where q_k, $k = 1, 2, 3$, are the cartesian coordinates of $\dot{\mathbf{R}}^3$. Note that

$$r := \left[\sum_{k=1}^{3} q_k^2\right]^{1/2} = \sum_{j=1}^{4} x_j^2 . \tag{2.5}$$

The fundamental vector field associated with the $U(1)$ action is determined to be

$$s_0(x) = (-x_2, x_1, -x_4, x_3)^{\mathrm{T}} \tag{2.6}$$

where the superscript T denotes the transpose. With respect to the standard inner product on $\mathbf{R}^4$, we can find vectors $s_k(x)$, $k = 1, 2, 3$, so that $s_0(x)$ and $s_k(x)$ may form an orthogonal basis in each co-tangent space $T_x^*(\dot{\mathbf{R}}^4)$:

$$\begin{aligned} &s_1(x) = (x_3, x_4, x_1, x_2)^{\mathrm{T}} \qquad s_2(x) = (x_4, -x_3, -x_2, x_1)^{\mathrm{T}} \\ &s_3(x) = (x_1, x_2, -x_3, -x_4)^{\mathrm{T}} . \end{aligned} \tag{2.7}$$

Now the $U(1)$ action (2.3) is lifted to a symplectic action on $T^*\dot{\mathbf{R}}^4$:

$$(x, y) \longmapsto (T(t)x, T(t)y) . \tag{2.8}$$

With this action is associated the momentum map $\Phi : T^*\dot{\mathbf{R}}^4 \longrightarrow u(1)^* \cong \mathbf{R}$, which is found to be

$$\Phi(x,y) = -x_2y_1 + x_1y_2 - x_4y_3 + x_3y_4 \tag{2.9}$$

where $u(1)^*$ is the dual to the Lie algebra $u(1)$ of $U(1)$. It is easy to see that Φ is $U(1)$-invariant. Following the Weinstein–Marsden reduction method (Abraham and Marsden 1978), we take a momentum space $\Phi^{-1}(\mu)$ for a fixed $\mu \in \mathbf{R}$ with $\mu \neq 0$, and form the reduced phase space $\Phi^{-1}(\mu)/U(1)$, which proves to be diffeomorphic with $T^*\dot{\mathbf{R}}^3 = \dot{\mathbf{R}}^3 \times \mathbf{R}^3$. The natural projection

$$\pi_\mu : \ \Phi^{-1}(\mu) \longrightarrow \Phi^{-1}(\mu)/U(1) \cong T^*\dot{\mathbf{R}}^3 \tag{2.10}$$

is realized by (2.4) together with

$$p_k = \frac{1}{2r} y \cdot s_k(x) \qquad k = 1,2,3 \tag{2.11}$$

where the dot denotes the inner product in $\mathbf{R}^4$. It is to be noted that for $\mu \neq 0$ $\Phi^{-1}(\mu)$ does not intersect the excluded points $\{0\} \times \mathbf{R}^4$ in $\mathbf{R}^4 \times \mathbf{R}^4$. Let ι_μ be the inclusion map $\Phi^{-1}(\mu) \longrightarrow T^*\dot{\mathbf{R}}^4$. The reduced symplectic form σ_μ is accordingly determined through

$$\pi_\mu^* \sigma_\mu = \iota_\mu^* \mathrm{d}\theta \tag{2.12}$$

and shown to be expressed as

$$\sigma_\mu = \sum_{k=1}^{3} \mathrm{d}p_k \wedge \mathrm{d}q_k + \mu\Omega$$

where

$$\Omega = \frac{-1}{2r^3}(q_1 \mathrm{d}q_2 \wedge \mathrm{d}q_3 + q_2 \mathrm{d}q_3 \wedge \mathrm{d}q_1 + q_3 \mathrm{d}q_1 \wedge \mathrm{d}q_2)\,. \tag{2.13}$$

Thus we have reduced the phase space $(T^*\dot{\mathbf{R}}^4, \mathrm{d}\theta)$ to $(T^*\dot{\mathbf{R}}^3, \sigma_\mu)$. See Iwai and Uwano (1986) for details.

3. The momentum map associated with $SU(2,2)$

To deal with $SU(2,2)$, it is convenient to introduce the complex vector space structure into $\mathbf{R}^4 \times \mathbf{R}^4$. For (x_j, y_j) given in the last section, set

$$\begin{aligned} z_1 &= x_1 + \mathrm{i}x_2 + x_3 + \mathrm{i}x_4 \qquad & z_2 &= -x_1 - \mathrm{i}x_2 + x_3 + \mathrm{i}x_4 \\ z_3 &= y_1 + \mathrm{i}y_2 + y_3 + \mathrm{i}y_4 \qquad & z_4 &= -y_1 - \mathrm{i}y_2 + y_3 + \mathrm{i}y_4 \end{aligned} \tag{3.1}$$

where $\mathrm{i} = \sqrt{-1}$, the imaginary unit. Further, let

$$u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \tfrac{1}{2}\left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - \mathrm{i} \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} \right) \qquad v := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \tfrac{1}{2}\left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \mathrm{i} \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} \right) \tag{3.2}$$

and

$$w := \begin{pmatrix} u \\ v \end{pmatrix} \tag{3.3}$$

that is, $w = (w_j)$ is defined by $w_k = u_k,\ w_{k+2} = v_k,\ k = 1, 2$. Then, the standard symplectic form given in (2.1) is put in the form

$$\mathrm{d}\theta = \frac{1}{2\mathrm{i}} \sum_{j,k=1}^{4} G_{jk} \mathrm{d}\overline{w}_j \wedge \mathrm{d}w_k \tag{3.4}$$

where

$$G := (G_{jk}) = \mathrm{diag}(1, 1, -1, -1) .$$

The $\mathrm{d}\theta$ is also expressed as

$$\mathrm{d}\theta = \frac{\mathrm{i}}{2} \mathrm{tr}(G\,\mathrm{d}w \wedge \mathrm{d}w^*) \tag{3.5}$$

where the superscript asterisk denotes the Hermitian conjugate, so that w^* is a row vector with complex conjugate components. In view of this, we introduce a 1-form

$$\Theta = \frac{i}{2} \mathrm{tr}(Gw\,\mathrm{d}w^*) \tag{3.6}$$

which satisfies $\mathrm{d}\Theta = \mathrm{d}\theta$. From now on, we consider the phase space $(\mathbf{C}^4, \mathrm{d}\Theta)$.

The $U(1)$ action given by (2.8) is now written simply as

$$w \longmapsto \mathrm{e}^{\mathrm{i}t} w \tag{3.7}$$

which is, of course, symplectic. Its infinitesimal generator is given by $\mathrm{i}w$ (in vector notation), so that the associated momentum map is determined by evaluating (3.6) for $\mathrm{i}w$:

$$\frac{\mathrm{i}}{2} \mathrm{tr}(Gw(\mathrm{i}w)^*) = \frac{1}{2}(|w_1|^2 + |w_2|^2 - |w_3|^2 - |w_4|^2) = \Phi . \tag{3.8}$$

Now we are to consider the group $SU(2, 2)$. By definition, a 4×4 complex matrix g is in $U(2, 2)$, if and only if

$$g^* G g = G \tag{3.9}$$

and further, a 4×4 complex matrix $\mathrm{i}\xi$ is in $u(2, 2)$, the Lie algebra of $U(2, 2)$, if and only if

$$G\xi = \xi^* G . \tag{3.10}$$

It is an easy matter to show that (3.9) and (3.10) are equivalent to

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \text{with} \qquad \begin{cases} A^*A - C^*C = \sigma_0 \\ B^*B - D^*D = -\sigma_0 \\ A^*B - C^*D = 0 \end{cases} \tag{3.11}$$

and to

$$\xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{with} \qquad a = a^*,\ d = d^*,\ c = -b^* \tag{3.12}$$

respectively, where $A, \ldots, D$ and $a, \ldots, d$ are 2×2 complex matrices and σ_0 is the 2×2 identity matrix. If one imposes conditions $\det g = 1$ and $\operatorname{tr} \xi = 0$ in addition, it turns out that $g \in SU(2,2)$ and $\mathrm{i}\xi \in su(2,2)$, respectively. Here we introduce the inner product in $u(2,2)$; for $\mathrm{i}\xi,\ \mathrm{i}\eta \in u(2,2)$, the inner product is defined to be

$$\gamma(\xi, \eta) = \tfrac{1}{2}\operatorname{tr}(\xi^* \eta)\,. \tag{3.13}$$

On this inner product, the dual space $u(2,2)^*$ is identified with $u(2,2)$.

It is an easy matter to see from (3.6) and (3.9) that $U(2,2)$ leaves Θ invariant, and hence is exact symplectic. For $\exp(\mathrm{i}t\xi) \in U(2,2)$, we denote its infinitesimal generator by $\mathrm{i}\xi_P$ with $P = \mathbf{C}^4$. Then the function $\Theta(\mathrm{i}\xi_P)$ determines the associated momentum map $\mathrm{i}J$ through

$$\Theta(\mathrm{i}\xi_P) = \tfrac{1}{2}\operatorname{tr}(ww^* G\xi) = \gamma(J(w), \xi)\,. \tag{3.14}$$

Put another way, $\mathrm{i}J : \mathbf{C}^4 \mapsto u(2,2)^* \cong u(2,2)$ is given by

$$J(w) = Gww^*\,. \tag{3.15}$$

It is an easy matter to show that J is Ad^*-equivariant:

$$J(gw) = \mathrm{Ad}_{g^{-1}*} J(w)\,. \tag{3.16}$$

According to the decomposition $u(2,2) \cong su(2,2) \oplus u(1)$, $J(w)$ is decomposed into $(Gww^* - \frac{1}{2}\Phi I_4) + \frac{1}{2}\Phi I_4$, where I_4 is the 4×4 identity matrix. We denote by $\mathrm{i}K(w)$ the $su(2,2)$ component of $\mathrm{i}J(w)$:

$$K(w) = Gww^* - \tfrac{1}{2}\Phi I_4\,. \tag{3.17}$$

It is now clear that for $g \in U(2,2)$

$$K(gw) = \mathrm{Ad}_{g^{-1}*} K(w) \qquad \Phi(gw) = \Phi(w)\,. \tag{3.18}$$

In the remainder of this section, we give the momentum map J in an explicit manner. In view of (3.12), we take a basis of $u(2,2)$, $\{\mathrm{i}e_\ell\}$, $\ell = 0, 1, \ldots, 15$, as follows:

$$\begin{aligned}
e_0 &= \tfrac{1}{2}\begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix} & e_1 &= \tfrac{1}{2}\begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \\
e_{j+1} &= \begin{pmatrix} \sigma_j & 0 \\ 0 & 0 \end{pmatrix} & e_{j+4} &= \begin{pmatrix} 0 & 0 \\ 0 & \sigma_j \end{pmatrix} \\
e_8 &= \tfrac{1}{2}\begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix} & e_{j+8} &= \tfrac{1}{2}\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \\
e_{12} &= \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} & e_{j+12} &= \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}
\end{aligned} \tag{3.19}$$

where σ_j, $j = 1,2,3$, are the Pauli spin matrices. Note that e_0 is a base of $u(1)$ and $e_1, \ldots, e_{15}$ form a basis of $su(2,2)$. The components of the momentum map J are given by

$$J_\ell(w) = \gamma(J(w), e_\ell) \qquad \ell = 0, 1, \ldots, 15 .$$

Calculation shows that

$$\begin{gathered} J_0(w) = \tfrac{1}{4}(\langle u,u\rangle - \langle v,v\rangle) = \tfrac{1}{2}\Phi \qquad J_1(w) = \tfrac{1}{4}(\langle u,u\rangle + \langle v,v\rangle) \\ J_{j+1}(w) = \tfrac{1}{2}\langle u, \sigma_j u\rangle \qquad J_{j+4}(w) = -\tfrac{1}{2}\langle v, \sigma_j v\rangle \qquad J_8(w) = \tfrac{1}{2}\mathrm{Re}\langle v, u\rangle \\ J_{j+8}(w) = \tfrac{1}{2}\mathrm{Re}\langle v, \sigma_j u\rangle \qquad J_{12}(w) = \tfrac{1}{2}\mathrm{Im}\langle v, u\rangle \qquad J_{j+12}(w) = \tfrac{1}{2}\mathrm{Im}\langle v, \sigma_j u\rangle \end{gathered} \tag{3.20}$$

where u, v are vectors given in (3.2) and $\langle\, ,\,\rangle$ denotes the Hermitian inner product in $\mathbf{C}^2$. Some of these functions are capable of dynamical understanding. Expressing $J_1(w)$ and $J_8(w)$ in the variables (x_j, y_j), one has

$$2J_1(w) = \tfrac{1}{2}\Big(\sum_j y_j^2 + \sum_j x_j^2\Big) \tag{3.21}$$

$$-2J_8(w) = \tfrac{1}{2}\Big(\sum_j y_j^2 - \sum_j x_j^2\Big) \tag{3.22}$$

$$(J_1(w) - J_8(w)) = \tfrac{1}{2}\sum_j y_j^2 \tag{3.23}$$

which are Hamiltonians for the harmonic oscillator, the repulsive oscillator, and a free particle, respectively. We note further that

$$2(J_1(w) + J_8(w)) = \sum_j x_j^2 = r . \tag{3.24}$$

4. Co-adjoint orbits of $SU(2,2)$

In this section, we show that the reduced phase space $T^*\dot{\mathbf{R}}^3$ is realized as a (co-)adjoint orbit of $SU(2,2)$, which fact will help us to understand that $SU(2,2)$ can be a dynamical group of a certain class of dynamical systems on $T^*\dot{\mathbf{R}}^3$.

For $\mu \neq 0$, we take a momentum space $\Phi^{-1}(\mu)$, which is defined by $\langle u,u\rangle - \langle v,v\rangle = 2\mu$. Making use of this quadratic form, we can show that $SU(2,2)$ acts transitively on $\Phi^{-1}(\mu)$. Take a point $w_0 = (\sqrt{2\mu}, 0, 0, 0)^{\mathrm{T}}$ in $\Phi^{-1}(\mu)$, where we have assumed that $\mu > 0$ without loss of generality. Then one has

$$\Phi^{-1}(\mu) = \{g w_0; g \in SU(2,2)\} \tag{4.1}$$

which is mapped, through the momentum map K, to

$$K(\Phi^{-1}(\mu)) = \{\mathrm{Ad}_{g^{-1*}} K(w_0); g \in SU(2,2)\} . \tag{4.2}$$

Here we have used (3.18). Equation (4.2) implies that $K(\Phi^{-1}(\mu))$ is a (co-)adjoint orbit in $su(2,2)$, which we denote by $\mathcal{O}_{K(w_0)}$.

We wish to show that the orbit $\mathcal{O}_{K(w_0)}$ is diffeomorphic to the reduced phase space $\Phi^{-1}(\mu)/U(1)$. Let ι_μ be the inclusion map $\Phi^{-1}(\mu) \longrightarrow \mathbf{C}^4$. We then consider the composite map

$$K \circ \iota_\mu : \Phi^{-1}(\mu) \longrightarrow \mathcal{O}_{K(w_0)} . \tag{4.3}$$

Suppose here that $K(w) = K(w')$ for $w,\ w' \in \Phi^{-1}(\mu)$. Hence one has $Gww^* = Gw'w'^*$, from which it follows that w and w' are related, for some $t \in \mathbf{R}$, by $w' = \mathrm{e}^{\mathrm{i}t} w$. This means that $[w] = [w']$ in $\Phi^{-1}(\mu)/U(1)$, where [] denotes the equivalence class. Thus $K \circ \iota_\mu$ induces the diffeomorphism

$$\widetilde{K}_\mu : \Phi^{-1}(\mu)/U(1) \longrightarrow \mathcal{O}_{K(w_0)} \tag{4.4}$$

which is defined by $\widetilde{K}_\mu([w]) = K \circ \iota_\mu(w)$. In other words, one has

$$\widetilde{K}_\mu \circ \pi_\mu = K \circ \iota_\mu . \tag{4.5}$$

We proceed to show further that $\widetilde{K}_\mu$ is a symplectomorphism. Any co-adjoint orbit is endowed with a symplectic form, called the Kirillov–Kostant–Souriau (KKS) form (Kirillov 1976). The KKS form on $\mathcal{O}_{K(w_0)}$ is defined, at $\nu \in \mathcal{O}_{K(w_0)}$, to be

$$\omega(\xi_Q, \eta_Q)(\nu) = -\frac{\mathrm{i}}{2}\mathrm{tr}(\nu^*[\xi, \eta]) \tag{4.6}$$

where $\mathrm{i}\xi_Q$ and $\mathrm{i}\eta_Q$ are the infinitesimal generators of the (co-)adjoint action of $\exp \mathrm{i}t\xi$ and $\exp \mathrm{i}t\eta$ on $Q := su(2,2)$, respectively, and $\mathrm{i}\nu \in Q$. From (3.18), the ξ_Q is related to ξ_P by

$$K_* \xi_P(w) = \xi_Q(\nu) \qquad \text{with} \qquad \nu = K(w), w \in P = \mathbf{C}^4 \tag{4.7}$$

where K_* denotes the differential of K. For ξ_P and η_P, and for $w \in \Phi^{-1}(\mu)$, we can prove, by using (4.6) and (4.7), that

$$\begin{aligned}(K \circ \iota_\mu)^* \omega(\xi_P(w), \eta_P(w)) &= \omega(\xi_Q(\nu), \eta_Q(\nu)) \\ &= \tfrac{1}{2}\mathrm{tr}(G\mathrm{d}w \wedge \mathrm{d}w^*)(\xi_P(w), \eta_P(w)) = \mathrm{d}\theta(\xi_P(w), \eta_P(w)) .\end{aligned}$$

Put another way,

$$(K \circ \iota_\mu)^* \omega = \iota_\mu^* \mathrm{d}\theta . \tag{4.8}$$

We are now ready to show that $\widetilde{K}_\mu$ is a symplectic map. In fact, from (2.12), (4.5) and (4.8), it follows that

$$\widetilde{K}_\mu^* \omega = \sigma_\mu . \tag{4.9}$$

Summing up the above discussion, we have the following.

Theorem 1. For $\mu \neq 0$, the reduced phase space $(\Phi^{-1}(\mu)/U(1), \sigma_\mu) = (T^*\dot{\mathbf{R}}^3, \sigma_\mu)$ is symplectomorphic with a (co-)adjoint orbit $(\mathcal{O}_{K(w_0)}, \omega)$ of $SU(2,2)$, where ω is the KKS form.

Since $U(1)$ and $SU(2,2)$ commute, $T^*\dot{\mathbf{R}}^3$ admits an $SU(2,2)$ action as a reduced phase space. Its infinitesimal generators are given as follows: the functions $J_\ell(w)$, $\ell = 1, 2, \ldots, 15$, are all invariant under the $U(1)$ action (3.7), and hence they project to functions $J_{\mu,\ell}$ on the reduced phase space through $J_\ell \circ \iota_\mu = J_{\mu,\ell} \circ \pi_\mu$. The Hamiltonian vector fields $X_{J_{\mu,\ell}}$ associated with $J_{\mu,\ell}$ turns out to be the infinitesimal generators of $SU(2,2)$ on $T^*\dot{\mathbf{R}}^3$, since they are related to the Hamiltonian vector fields X_{J_ℓ} by

$$\pi_{\mu *} X_{J_\ell}(\iota_\mu(w)) = X_{J_{\mu,\ell}}(\pi_\mu(w)) .$$

Hence, if a linear combination $F_\mu = \sum_\ell c_\ell J_{\mu,\ell}$ is taken to be a Hamiltonian, the system $(T^*\dot{\mathbf{R}}^3, \sigma_\mu, F_\mu)$ has $SU(2,2)$ as a dymanical group.

In concluding this section, we make some remarks on the Kepler problem and (co-)adjoint orbits of $SU(2,2)$. For the Kepler problem, a particular (co-)adjoint orbit of $SO_0(2,4)$ is used, which is symplectomorphic with T^+S^3, the set of non-zero cotangent vectors to S^3. Note here the isomorphism $SU(2,2)/\mathbf{Z}_2 \cong SO_0(2,4)$ with $\mathbf{Z}_2 = \{I_4, -I_4\}$. This orbit is well described in Guillemin and Sternberg (1977, 1990) and Kummer (1982). Kummer (1982) showed that if the domain of Φ is restricted to $\mathbf{C}^2 - \{0\}$, the reduced phase space $\Phi^{-1}(0)/U(1)$ is diffeomorphic with T^+S^3. He also made extensive use of (3.20) to show the isomorphism of T^+S^3 with the particular orbits mentioned above—see also Cordani (1986, 1987).

5. The MIC–Kepler problem and its symmetry groups

In a series of papers, Iwai and Uwano (1986, 1988, 1991, 1991) have defined and analyzed the MIC–Kepler problem as an extension of the usual Kepler problem. According to their results, the symmetry groups of the MIC–Kepler problem are $SO(4)$, $SO_0(3,1)$, and $E(3)$, depending on whether the energy is negative, positive, or zero. In this section, we show that these groups are all subgroups of $SU(2,2)$. Strictly speaking, the double cover of the respective groups, $SU(2) \times SU(2)$, $SL(2,\mathbf{C})$, and $SU(2) \ltimes \mathbf{R}^3$, are realized as subgroups of $SU(2,2)$.

To start with, we give the definition of the MIC–Kepler problem. Let H_c be the Hamiltonian of the conformal Kepler problem on $(T^*\dot{\mathbf{R}}^4, \mathrm{d}\theta)$:

$$H_c = \frac{1}{2}\left(\frac{1}{4r}\sum_{j=1}^{4} y_j^2\right) - \frac{\kappa}{r} \qquad \kappa > 0; \mathrm{const}. \tag{5.1}$$

After the reduction method in section 2, the conformal Kepler problem $(T^*\dot{\mathbf{R}}^4, \mathrm{d}\theta, H_c)$ is reduced to the MIC–Kepler problem $(T^*\dot{\mathbf{R}}^3, \sigma_\mu, H_\mu)$ by definition, where H_μ is determined by $H_c \circ \iota_\mu = H_\mu \circ \pi_\mu$ and turns out to be expressed as

$$H_\mu = \frac{1}{2}\sum_{k=1}^{3} p_k^2 + \frac{\mu^2}{8r^2} - \frac{\kappa}{r} . \tag{5.2}$$

In order to study the symmetry of the MIC–Kepler problem, we can use the reduction method. Our technique is to introduce the Hamiltonians

$$A_\lambda = \frac{1}{2}\sum_j y_j^2 + \frac{\lambda^2}{2}\sum_j x_j^2 \tag{5.3}$$

$$R_\lambda = \frac{1}{2}\sum_j y_j^2 - \frac{\lambda^2}{2}\sum_j x_j^2 \tag{5.4}$$

$$F = \tfrac{1}{2}\sum_j y_j^2 \tag{5.5}$$

where λ is a positive parameter. Then these Hamiltonians are related to H_c by

$$4r\left(H_c + \frac{\lambda^2}{8}\right) = A_\lambda - 4\kappa \tag{5.6}$$

$$4r\left(H_c - \frac{\lambda^2}{8}\right) = R_\lambda - 4\kappa \tag{5.7}$$

$$4rH_c = F - 4\kappa\,. \tag{5.8}$$

Hence, the energy manifold $H_c^{-1}(E)$ of the conformal Kepler problem coincides with that of the harmonic oscillator, the repulsive oscillator, or a free particle, depending on whether E is negative, positive, or zero. Moreover, the Hamiltonian flow of X_{H_c} coincides, up to parametrization, with that of X_{A_λ},X_{R_λ}, or X_F, according to whether E is negative, positive, or zero. In fact, one has $4rX_{H_c} = X_{A_\lambda}, X_{R_\lambda}$, or X_F on $H_c^{-1}(E)$, according to $E = -\lambda^2/8, \lambda^2/8$, or 0. From the definition of H_μ and the reduction procedure, the energy manifold $H_\mu^{-1}(E)$ of the MIC–Kepler problem is given by $H_c^{-1}(E) \cap \Phi^{-1}(\mu)/U(1)$, and therefore proves to be expressed as

$$H_\mu^{-1}(E) = A_\lambda^{-1}(4\kappa) \cap \Phi^{-1}(\mu)/U(1) \tag{5.9}$$

$$H_\mu^{-1}(E) = R_\lambda^{-1}(4\kappa) \cap \Phi^{-1}(\mu)/U(1) \tag{5.10}$$

$$H_\mu^{-1}(E) = F^{-1}(4\kappa) \cap \Phi^{-1}(\mu)/U(1) \tag{5.11}$$

depending on whether $E = -\lambda^2/8,\ \lambda^2/8$, or 0.

The symmetry groups for the MIC–Kepler problem are those groups which act on the respective energy manifolds $H_\mu^{-1}(E)$. Since the topology of the respective energy manifolds is independent of λ, we fix $\lambda = 1$ below. From (5.9) the symmetry group for $E < 0$ must be a subgroup of $SU(2,2)$ which preserves A_1 and Φ. Since $SU(2,2)$ already preserves Φ, that subgroup should preserve $A_1 = 2J_1$, so that it must commute with the one-parameter subgroup generated by X_{A_1}, which is put in the form

$$\begin{pmatrix} e^{it}\sigma_0 & 0 \\ 0 & e^{-it}\sigma_0 \end{pmatrix}. \tag{5.12}$$

The subgroup commutative with this is shown, after a calculation along with (3.11) and (5.12), to take the form

$$G^- := \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}; A, B \in SU(2) \right\}. \tag{5.13}$$

Here we have restricted the subgroup so as not to include (5.12). Thus we have obtained the symmetry group G^- isomorphic to $SU(2) \times SU(2)$. Since (3.7) and (5.13) commute, the group G^- projects to act on the energy manifold $H_\mu^{-1}(E)$, as is seen from (5.9).

In the case of $E > 0$ and of $E = 0$, the same reasoning applies. For $E > 0$, the symmetry group should preserve R_1 and Φ. The one-parameter subgroup generated by $X_{R_1} = -2J_8$ is given by

$$\begin{pmatrix} \sigma_0 \cosh t & -\mathrm{i}\sigma_0 \sinh t \\ \mathrm{i}\sigma_0 \sinh t & \sigma_0 \cosh t \end{pmatrix} \tag{5.14}$$

so that the subgroup commutative with this can be shown to be expressed as

$$G^+ := \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix}; A^*A - B^*B = \sigma_0, A^*B + B^*A = 0, \det(A + \mathrm{i}B) = 1 \right\}. \tag{5.15}$$

The subgroup (5.14) has been excluded from (5.15). The group (5.15) is shown to be isomorphic with $SL(2, \mathbf{C})$ having elements of the form $A + \mathrm{i}B$.

For $E = 0$, the symmetry group must preserve $F = J_1 - J_8$ and Φ, and therefore commutes with the one-parameter subgroup

$$\begin{pmatrix} (1 + \mathrm{i}t)\sigma_0 & -\mathrm{i}t\sigma_0 \\ \mathrm{i}t\sigma_0 & (1 - \mathrm{i}t)\sigma_0 \end{pmatrix} \tag{5.16}$$

which is generated by X_F. A calculation gives a subgroup commutative with (5.16) as follows:

$$G^0 := \left\{ \begin{pmatrix} A & B \\ -B & A + 2B \end{pmatrix}; \begin{matrix} A^*A - B^*B = \sigma_0, & \det(A + B) = 1 \\ A^*B + B^*(A + 2B) = 0, & \mathrm{tr}(A + B)^{-1}B = 0 \end{matrix} \right\}. \tag{5.17}$$

The subgroup (5.16) is not included in (5.17). In order to find what group this subgroup is isomorphic with, we introduce a mapping A_S, which is defined for g in G^0 to be

$$A_S g = SgS^{-1} \qquad \text{with} \qquad S = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_0 & -\sigma_0 \\ \sigma_0 & \sigma_0 \end{pmatrix}. \tag{5.18}$$

Then (5.17) becomes

$$G_S^0 := \left\{ \begin{pmatrix} A + B & 0 \\ -2B & A + B \end{pmatrix}; A + B \in SU(2),\ (A + B)^{-1}B \in su(2) \right\}. \tag{5.19}$$

From this it follows that G_S^0 is isomorphic with a semi-direct product group $SU(2) \ltimes \mathbf{R}^3$. In fact, the isomorphism $G_S^0 \to SU(2) \ltimes \mathbf{R}^3$ is given by

$$\begin{pmatrix} A + B & 0 \\ -2B & A + B \end{pmatrix} \longmapsto \left(A + B, (A + B)^{-1}B\right) \in SU(2) \times \underline{su(2)}$$

where $\underline{su(2)}$ denotes the underlying vector space of $su(2)$ and is isomorphic with $\mathbf{R}^3$. Summing up the above discussion, we have the following theorem.

Theorem 2. All the symmetry groups for the MIC–Kepler problem are realized as subgroups of $SU(2,2)$, which are G^-, G^+, G^0 defined by (5.13), (5.15) and (5.17) for negative, positive, and zero energies, respectively. They are isomorphic with $SU(2) \times SU(2)$, $SL(2,\mathbf{C})$, and $SU(2) \ltimes \mathbf{R}^3$, respectively.

Thus we have found all the symmetry groups of the MIC–Kepler problem. It is to be noted that since the symmetry groups are subgroups of $SU(2,2)$ and since the respective energy manifolds $H_\mu^{-1}(E)$ are regarded as submanifolds of $\mathcal{O}_{K(w_0)}$ from theorem 1, the action of the symmetry group is considered as a (co-)adjoint action, so that $SU(2) \times SU(2)/\mathbf{Z}_2$, $SL(2,\mathbf{C})/\mathbf{Z}_2$, and $SU(2) \ltimes \mathbf{R}^3/\mathbf{Z}_2$ with $\mathbf{Z}_2 = \{I_4, -I_4\}$ should be taken to be the symmetry groups, which are isomorphic with $SO(4)$, $SO_0(1,3)$, and $E(3)$, respectively.

The Hamiltonian H_μ itself is not a momentum function for $SU(2,2)$, but is a function of momentum functions; in fact, from (3.23), (3.24), and (5.1), H_c is expressible as a function of J_1 and J_8, so that H_μ turns out to be a function of $J_{\mu,1}$ and $J_{\mu,8}$. Further, all the associated functions, A_1, R_1, and F, are momentum functions for $SU(2,2)$, which were given in (3.21), (3.22), and (3.23), respectively. In view of all these facts, we may interpret $SU(2,2)$ as a dynamical group for the MIC–Kepler problem.

6. Isoenergetic orbit spaces

According to whether E is positive, negative, or zero, the action of (5.12), (5.14), or (5.16) provides the Hamiltonian flows for H_c within a change of parameters. Further, each action commutes with the $U(1)$ action (3.7), so that those Hamiltonian flows project to respective energy manifolds $H_\mu^{-1}(E)$ to determine the Hamiltonian flows for H_μ. We denote by G_t the Hamiltonian flow on $H_\mu^{-1}(E)$.

The isoenergetic orbit space for the MIC–Kepler problem then can be defined to be $H_\mu^{-1}(E)/G_t$. In this section, we will show that the isoenergetic orbit spaces are (co-)adjoint orbits of the respective symmetry groups.

We start with the case of $E < 0$. For the group G^- given by (5.13), we denote its Lie algebra by $\mathcal{G}^-$, which is isomorphic with $su(2) \oplus su(2)$:

$$\mathcal{G}^- := \left\{ \mathrm{i}\xi = \begin{pmatrix} \mathrm{i}\xi_1 & 0 \\ 0 & \mathrm{i}\xi_2 \end{pmatrix} ; \mathrm{i}\xi_k \in su(2) \quad k = 1,2 \right\}. \tag{6.1}$$

Then the inner product (3.13) for $\xi = \xi_1 \oplus \xi_2$ and $\eta = \eta_1 \oplus \eta_2$ is put in the form

$$\gamma(\xi, \eta) = \tfrac{1}{2}\mathrm{tr}(\xi_1^* \eta_1) + \tfrac{1}{2}\mathrm{tr}(\xi_2^* \eta_2) \tag{6.2}$$

which induces the inner product on $\mathcal{G}^-$, and thereby $\mathcal{G}^-$ and its dual are identified. Further, the function (3.14) turns into

$$\tfrac{1}{2}\mathrm{tr}(ww^* G\xi) = \tfrac{1}{2}\mathrm{tr}(uu^*\xi_1 - vv^*\xi_2). \tag{6.3}$$

From (6.2) and (6.3), the momentum map associated with $G^- \cong SU(2) \times SU(2)$,

$$\mathrm{i}K^- = \mathrm{i}K_\mathrm{L}^- \oplus \mathrm{i}K_\mathrm{R}^- : \mathbf{C}^4 = \mathbf{C}^2 \times \mathbf{C}^2 \longrightarrow \mathcal{G}^- \cong su(2) \oplus su(2)$$

is found to be

$$K^-(w) = \begin{pmatrix} uu^* & 0 \\ 0 & -vv^* \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \langle u,u\rangle\sigma_0 & 0 \\ 0 & -\langle v,v\rangle\sigma_0 \end{pmatrix} \tag{6.4}$$

together with

$$K_{\mathrm{L}}^-(u) = uu^* - \tfrac{1}{2}\langle u,u\rangle\sigma_0 \qquad K_{\mathrm{R}}^-(v) = -vv^* + \tfrac{1}{2}\langle v,v\rangle\sigma_0 .$$

Clearly, $K^- = K_{\mathrm{L}}^- \oplus K_{\mathrm{R}}^-$ is Ad-equivariant.

We are ready to study the isoenergetic orbit space $H_\mu^{-1}(E)/G_{\mathrm{t}}$ for $E < 0$. From (5.9), $H_\mu^{-1}(E)/G_{\mathrm{t}}$ turns out to be given by $A_1^{-1}(4\kappa) \cap \Phi^{-1}(\mu)/U(1) \times U(1)$, where $U(1) \times U(1)$ is the product group of (5.12) and (3.7). Note here that A_1 and Φ are the momentum maps associated with (5.12) and (3.7), respectively. Hence this isoenergetic orbit space is regarded as the reduced phase space by the group $U(1) \times U(1)$. We denote the natural projection and the isoenergetic orbit space by π_μ^- and M_μ^-, respectively:

$$\pi_\mu^- : A_1^{-1}(4\kappa) \cap \Phi^{-1}(\mu) \longrightarrow M_\mu^- := A_1^{-1}(4\kappa) \cap \Phi^{-1}(\mu)/U(1) \times U(1) . \tag{6.5}$$

As is easily shown, the manifold $A_1^{-1}(4\kappa) \cap \Phi^{-1}(\mu)$ is determined by the conditions $\langle u,u\rangle = 4\kappa + \mu$ and $\langle v,v\rangle = 4\kappa - \mu$, where μ and κ *must satisfy the condition that* $4\kappa - |\mu| \geqslant 0$. If $4\kappa - |\mu| > 0$, $A_1^{-1}(4\kappa) \cap \Phi^{-1}(\mu)$ is diffeomorphic with $S^3 \times S^3$, so that it admits a transitive action of $SU(2) \times SU(2)$. Therefore one has

$$A_1^{-1}(4\kappa) \cap \Phi^{-1}(\mu) = \{(gu_0, hv_0); (g,h) \in SU(2) \times SU(2)\}$$

where $u_0 = (\sqrt{4\kappa + \mu}, 0)^{\mathrm{T}}$ and $v_0 = (\sqrt{4\kappa - \mu}, 0)^{\mathrm{T}}$. Applying the momentum map $K^- = K_{\mathrm{L}}^- \oplus K_{\mathrm{R}}^-$ to this, we obtain

$$K^-(A_1^{-1}(4\kappa)\cap\Phi^{-1}(\mu)) = \{\mathrm{Ad}_g K_{\mathrm{L}}^-(u_0) \oplus \mathrm{Ad}_h K_{\mathrm{R}}^-(v_0);\ (g,h) \in SU(2)\times SU(2)\} .$$

This implies that the image of K^- is an adjoint orbit in $su(2) \oplus su(2)$, which we denote by $\mathcal{O}^-$.

Letting ι_μ^- be the inclusion map: $A_1^{-1}(4\kappa) \cap \Phi^{-1}(\mu) \longrightarrow \mathbf{C}^2 \times \mathbf{C}^2$, we consider, like (4.3), the composite map

$$K^- \circ \iota_\mu^- : A_1^{-1}(4\kappa) \cap \Phi^{-1}(\mu) \longrightarrow \mathcal{O}^- . \tag{6.6}$$

Suppose here that the map (6.6) has the same image for (u', v') and (u, v), so that $u'u'^* = uu^*$ and $v'v'^* = vv^*$. Then one has $u' = \mathrm{e}^{\mathrm{i}t}u$, and $v' = \mathrm{e}^{\mathrm{i}s}v$ for some $t, s \in \mathbf{R}$. Let us here be reminded that the $U(1)$ action (3.7) can be written as $u \mapsto \mathrm{e}^{\mathrm{i}\tau}u$, $v \mapsto \mathrm{e}^{\mathrm{i}\tau}v$, and that the action of (5.12) is expressed as $u \mapsto \mathrm{e}^{\mathrm{i}\sigma}u$, $v \mapsto \mathrm{e}^{-\mathrm{i}\sigma}v$. Hence the $U(1) \times U(1)$ action is put in the form

$$u \mapsto \mathrm{e}^{\mathrm{i}(\tau+\sigma)}u \qquad v \mapsto \mathrm{e}^{\mathrm{i}(\tau-\sigma)}v .$$

Changing the parameters by $\sigma + \tau = t, \tau - \sigma = s$, one has $u \mapsto \mathrm{e}^{\mathrm{i}t}u$ and $v \mapsto \mathrm{e}^{\mathrm{i}s}v$. From this it turns out that $K_{\mathrm{L}}^-(u') = K_{\mathrm{L}}^-(u)$ and $K_{\mathrm{R}}^-(v') = K_{\mathrm{R}}^-(v)$ imply that

$[(u',v')] = [(u,v)]$, where [ ] denotes the equivalence class by the $U(1)\times U(1)$ action. Thus one has a diffeomorphism

$$\widetilde{K}_\mu^- : A_1^{-1}(4\kappa)\cap\Phi^{-1}(\mu)/U(1)\times U(1)\longrightarrow \mathcal{O}^- \tag{6.7}$$

which is determined by $\widetilde{K}_\mu^-([(u,v)]) = K^-\circ\iota_\mu(u,v)$. Put another way,

$$\widetilde{K}_\mu^-\circ\pi_\mu^- = K^-\circ\iota_\mu^- . \tag{6.8}$$

We can prove that $\widetilde{K}_\mu^-$ is symplectomorphic. To show this, we follow the same method as in section 4. First we note that the standard symplectic form (3.5) can be written as

$$\mathrm{d}\theta = \frac{\mathrm{i}}{2}\mathrm{tr}(\mathrm{d}u\wedge\mathrm{d}u^* - \mathrm{d}v\wedge\mathrm{d}v^*) \tag{6.9}$$

and that the KKS form on the (co-)adjoint orbit $\mathcal{O}^-$ is defined at $\nu\in\mathcal{O}^-$ to be

$$\omega^-(\xi_Q,\eta_Q)(\nu) = -\frac{\mathrm{i}}{2}\mathrm{tr}(\nu_1^*[\xi_1,\eta_1]) - \frac{\mathrm{i}}{2}\mathrm{tr}(\nu_2^*[\xi_2,\eta_2]) \tag{6.10}$$

where $\xi=\xi_1\oplus\xi_2$, $\eta=\eta_1\oplus\eta_2$, $\nu=\nu_1\oplus\nu_2$ with $\xi_1,\ldots,\nu_2$ being 2×2 traceless Hermitian matrices. Definition (6.10) is the restriction of (4.6) to $\mathcal{O}^-$. Then from (6.4), (6.9), and (6.10) together with $\nu = K^-(w)$, we can prove, like (4.8), that

$$(K^-\circ\iota_\mu^-)^*\omega^- = (\iota_\mu^-)^*\mathrm{d}\theta . \tag{6.11}$$

Equations (6.8) and (6.11) are put together to give

$$(\widetilde{K}_\mu^-)^*\omega^- = \sigma_\mu^- \tag{6.12}$$

where σ_μ^- is the reduced symplectic form which is defined on the reduced phase space M_μ^- through $(\pi_\mu^-)^*\sigma_\mu^- = (\iota_\mu^-)^*\mathrm{d}\theta$.

We proceed to the case of $E>0$. The Lie algebra of the group (5.15) is given by

$$\mathcal{G}^+ := \left\{\mathrm{i}\xi = \begin{pmatrix}\mathrm{i}\xi_1 & \mathrm{i}\xi_2\\ -\mathrm{i}\xi_2 & \mathrm{i}\xi_1\end{pmatrix}; \mathrm{i}\xi_k\in su(2), k=1,2\right\} \tag{6.13}$$

which is isomorphic with $s\ell(2,\mathrm{C})$ by the map $\mathrm{i}\xi\mapsto\mathrm{i}(\xi_1+\xi_2)$. For $\mathrm{i}\xi$ and $\mathrm{i}\eta$ in $\mathcal{G}^+$, the inner product (3.13) is restricted to induce

$$\gamma(\xi,\eta) = \mathrm{tr}(\xi_1^*\eta_1 + \xi_2^*\eta_2) \tag{6.14}$$

which equips $\mathcal{G}^+$ with an inner product, and thereby $\mathcal{G}^+$ and its dual are identified. Further, for $\mathrm{i}\xi\in\mathcal{G}^+$, the function (3.14) reduces to

$$\tfrac{1}{2}\mathrm{tr}(ww^*G\xi) = \tfrac{1}{2}\mathrm{tr}((uu^*-vv^*)\xi_1 + (uv^*+vu^*)\xi_2) . \tag{6.15}$$

Hence, the momentum map associated with the group $G^+ \cong SL(2, \mathbf{C})$,

$$\mathrm{i}K^+ : \mathbf{C}^4 \longmapsto \mathcal{G}^+ \cong s\ell(2, \mathbf{C})$$

is expressed as

$$K^+(w) = \tfrac{1}{2}\begin{pmatrix} U & V \\ -V & U \end{pmatrix} - \tfrac{1}{4}\begin{pmatrix} (\mathrm{tr}U)\sigma_0 & (\mathrm{tr}V)\sigma_0 \\ -(\mathrm{tr}V)\sigma_0 & (\mathrm{tr}U)\sigma_0 \end{pmatrix} \tag{6.16}$$

where

$$U = uu^* - vv^* \qquad V = uv^* + vu^* .$$

As is easily shown, K^+ is co-adjoint-equivariant for G^+ and invariant for $U(1)$:

$$K^+(gw) = \mathrm{Ad}_{g^{-1*}} K^+(w) \qquad K^+(\mathrm{e}^{\mathrm{i}t} w) = K^+(w) . \tag{6.17}$$

We turn to the isoenergetic orbit space. From (5.10) it follows that $H_\mu^{-1}(E)/G_t$ for $E > 0$ is given by $R_1^{-1}(4\kappa) \cap \Phi^{-1}(\mu)/\mathbf{R} \times U(1)$, where $\mathbf{R} \times U(1)$ is the product group of (5.14) and (3.7). Note here that $-R_1$ and Φ are the momentum maps associated with (5.14) and (3.7), respectively. Hence the isoenergetic orbit space is viewed as the reduced phase space by the group $\mathbf{R} \times U(1)$. We denote the natural projection and the isoenergetic orbit space by π_μ^+ and M_μ^+, respectively:

$$\pi_\mu^+ : R_1^{-1}(4\kappa) \cap \Phi^{-1}(\mu) \longrightarrow M_\mu^+ := R_1^{-1}(4\kappa) \cap \Phi^{-1}(\mu)/\mathbf{R} \times U(1) . \tag{6.18}$$

We know already that the group G^+ given by (5.15) acts on $R_1^{-1}(4\kappa) \cap \Phi^{-1}(\mu)$, since G^+ preserves R_1 and Φ from the definition. We now show that $U(1) \times G^+$ acts transitively on that manifold. Take a fixed point (u_0, v_0) and an arbitrary point (u, v) of $R_1^{-1}(4\kappa) \cap \Phi^{-1}(\mu)$, where

$$u_0 = \sqrt{2}\begin{pmatrix} \sqrt{\mu} + \mathrm{i}\sqrt{2\kappa} \\ 0 \end{pmatrix} \qquad v_0 = \sqrt{2}\begin{pmatrix} -\mathrm{i}\sqrt{2\kappa} \\ 0 \end{pmatrix} .$$

For (u, v), we set

$$u' = \frac{1}{\sqrt{2\mu^2 + 32\kappa^2}}((\sqrt{\mu} + \mathrm{i}\sqrt{2\kappa})u - \mathrm{i}\sqrt{2\kappa}v),$$

$$v' = \frac{1}{\sqrt{2\mu^2 + 32\kappa^2}}((\mathrm{i}\sqrt{2\kappa}u + (\sqrt{\mu} + \mathrm{i}\sqrt{2\kappa})v).$$

Then u' and v' satisfy

$$\langle u', u' \rangle - \langle v', v' \rangle = 1 \qquad \langle u', v' \rangle + \langle v', u' \rangle = 0 .$$

Let A and B be two 2×2 matrices which have u' and v' as the first-column vector, respectively, and suitably chosen second-column vectors so as to form an element of the group (5.15). It then follows that

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \frac{\mu - 4\kappa + 2\mathrm{i}\sqrt{2\mu\kappa}}{\sqrt{\mu^2 + 16\kappa^2}}\begin{pmatrix} u \\ v \end{pmatrix} . \tag{6.19}$$

The coefficient of the right-hand side of (6.19) has the absolute value one. This equation is what we wanted to show. From (6.19), we obtain

$$R_1^{-1}(4\kappa) \cap \Phi^{-1}(\mu) = \{e^{it} g w_0; g \in G^+\}$$

where $w_0 = \binom{u_0}{v_0}$. Applying the momentum map K^+ to this set and using (6.17), we are led to

$$K^+(R_1^{-1}(4\kappa) \cap \Phi^{-1}(\mu)) = \{\mathrm{Ad}_{g^{-1}*} K^+(w_0); g \in G^+\}. \tag{6.20}$$

Equation (6.20) implies that the image $K^+(R_1^{-1}(4\kappa) \cap \Phi^{-1}(\mu))$ is a (co-)adjoint orbit of $G^+ \cong SL(2, \mathbf{C})$, which we denote by $\mathcal{O}^+$.

Letting ι_μ^+ be the inclusion map $R_1^{-1}(4\kappa) \cap \Phi^{-1}(\mu) \to \mathbf{C}^4$, we consider the composite map

$$K^+ \circ \iota_\mu^+ : R_1^{-1}(4\kappa) \cap \Phi^{-1}(\mu) \longrightarrow \mathcal{O}^+. \tag{6.21}$$

Suppose here that $K^+ \circ \iota_\mu^+(u, v) = K^+ \circ \iota_\mu^+(u', v')$, so that $uu^* - vv^* = u'u'^* - v'v'^*$ and $uv^* + vu^* = u'v'^* + v'u'^*$. Then it turns out that (u, v) and (u', v') must lie in the same orbit of the product of the groups (3.7) and (5.14). Hence $K^+ \circ \iota_\mu^+$ induces a diffeomorphism

$$\widetilde{K}_\mu^+ : R_1^{-1}(4\kappa) \cap \Phi^{-1}(\mu)/\mathbf{R} \times U(1) \longrightarrow \mathcal{O}^+ \tag{6.22}$$

which satisfies

$$\widetilde{K}_\mu^+ \circ \pi_\mu^+ = K^+ \circ \iota_\mu^+. \tag{6.23}$$

In the same manner as that for $\widetilde{K}_\mu^-$, we can prove that $\widetilde{K}_\mu^+$ is symplectomorphic. Applied for $i\nu$, $i\xi$, $i\eta \in \mathcal{G}^+$, (4.6) provides the KKS form on the co-adjoint orbit $\mathcal{O}^+$, which is expressed, at $\nu \in \mathcal{O}^+$, as

$$\omega^+(\xi_Q, \eta_Q)(\nu) = -i \operatorname{tr}(\nu_1^*([\xi_1, \eta_1] - [\xi_2, \eta_2]) + \nu_2^*([\xi_1, \eta_2] + [\xi_2, \eta_1])). \tag{6.24}$$

Then, by use of (6.9) and (6.24) together with $\nu = K^+(w)$, we can show that

$$(K^+ \circ \iota_\mu^+)^* \omega^+ = (\iota_\mu^+)^* d\theta. \tag{6.25}$$

Equations (6.23) and (6.25) are combined to yield

$$(\widetilde{K}_\mu^+)^* \omega^+ = \sigma_\mu^+ \tag{6.26}$$

where σ_μ^+ is the reduced symplectic form on the reduced phase space M_μ^+, which form is defined through $(\pi_\mu^+)^* \sigma_\mu^+ = (\iota_\mu^+)^* d\theta$.

We come to the last case of $E = 0$. According to (5.18), it is convenient to introduce the new vectors

$$u_s = \frac{1}{\sqrt{2}}(u - v) \qquad v_s = \frac{1}{\sqrt{2}}(u + v). \tag{6.27}$$

Put another way, $w_s := \binom{u_s}{v_s} = Sw$. Then the 1-form Θ given by (3.6) is rewritten as

$$\Theta = \frac{\mathrm{i}}{2}\mathrm{tr}(G_s w_s \,\mathrm{d}(w_s)^*) = \frac{\mathrm{i}}{2}\mathrm{tr}(v_s \,\mathrm{d}u_s^* + u_s \,\mathrm{d}v_s^*) \tag{6.28}$$

where

$$G_s := A_S \, G = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}.$$

The Lie algebra of (5.19) is given by

$$\mathcal{G}_S^0 := \left\{ \mathrm{i}\xi = \begin{pmatrix} \mathrm{i}(\xi_1+\xi_2) & 0 \\ -2\mathrm{i}\xi_2 & \mathrm{i}(\xi_1+\xi_2) \end{pmatrix}; \mathrm{i}\xi_k \in su(2) \quad k = 1,2 \right\} \tag{6.29}$$

which is isomorphic with a semi-direct sum of $su(2)$ and $\underline{su(2)}$, $su(2) \oplus_s \underline{su(2)}$, where $\underline{su(2)}$ denotes the underlying vector space of $su(2)$; the isomorphism is given by

$$\begin{pmatrix} \mathrm{i}(\xi_1+\xi_2) & 0 \\ -2\mathrm{i}\xi_2 & \mathrm{i}(\xi_1+\xi_2) \end{pmatrix} \longmapsto (\mathrm{i}(\xi_1+\xi_2),\ \mathrm{i}\xi_2) \in su(2) \oplus_s \underline{su(2)}.$$

For $\mathrm{i}\xi \in \mathcal{G}_S^0$, the function (3.14) takes, in turn, the form

$$\tfrac{1}{2}\mathrm{tr}(w_s w_s^* G_s \xi) = \tfrac{1}{2}\mathrm{tr}\left((u_s v_s^* + v_s u_s^*)(\xi_1+\xi_2) - 2u_s u_s^* \xi_2\right). \tag{6.30}$$

The momentum map associated with G_S^0 should be determined through (6.30). To this end, we have to think of the inner product on $\mathcal{G}_S^0$. For $\mathrm{i}\xi,\ \mathrm{i}\eta \in \mathcal{G}_S^0$, the inner product (3.11) is reduced to

$$\gamma(\xi,\eta) = \tfrac{1}{2}\mathrm{tr}\left((\xi_1+\xi_2)^*(\eta_1+\eta_2) + 2\xi_2^*\eta_2\right).$$

While this equips $\mathcal{G}_s^0$ with an inner product, it is not suitable for our purpose. We choose to take the inner product defined, for $\mathrm{i}\xi,\ \mathrm{i}\eta \in \mathcal{G}_S^0$, to be

$$\tfrac{1}{2}\mathrm{tr}(\xi G_s \eta) = -\mathrm{tr}\left((\xi_1+\xi_2)\eta_2 + \xi_2(\eta_1+\eta_2)\right). \tag{6.31}$$

With respect to this inner product, $\mathcal{G}_S^0$ and its dual are identified. Then, from (6.30) and (6.31), the momentum map associated with $G_S^0 \cong SU(2) \ltimes \mathbf{R}^3$,

$$K^0 : \ \mathbf{C}^4 \longrightarrow \mathcal{G}_S^0 \cong su(2) \oplus_s \underline{su(2)}$$

is expressed as

$$K^0(w_s) = \begin{pmatrix} W_s & 0 \\ U_s & W_s \end{pmatrix} - \tfrac{1}{2}\begin{pmatrix} (\mathrm{tr}W_s)\sigma_0 & 0 \\ (\mathrm{tr}U_s)\sigma_0 & (\mathrm{tr}W_s)\sigma_0 \end{pmatrix} \tag{6.32}$$

where

$$W_s = u_s u_s^* \qquad U_s = u_s v_s^* + v_s u_s^*.$$

This momentum map is shown to be Ad-equivariant for G_S^0 and invariant for $U(1)$:

$$K^0(gw_s) = \mathrm{Ad}_g K^0(w_s) \qquad K^0(\mathrm{e}^{\mathrm{i}t} w_s) = K^0(w_s) . \tag{6.33}$$

Now the isoenergetic orbit space $H_\mu^{-1}(E)/G_t$ for $E = 0$ is shown from (5.11) to be $F^{-1}(4\kappa) \cap \Phi^{-1}(\mu)/\mathbf{R} \times U(1)$, where $\mathbf{R} \times U(1)$ is the product group of (5.16) and (3.7), each of which is transformed into

$$\begin{pmatrix} \sigma_0 & 0 \\ 2\mathrm{i}t\sigma_0 & \sigma_0 \end{pmatrix} \qquad \text{and} \qquad \mathrm{e}^{\mathrm{i}t} \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix} \tag{6.34}$$

respectively, with respect to the variable w_s. We here note that

$$F = \tfrac{1}{2}\langle u_s, u_s \rangle \qquad \Phi = \tfrac{1}{2}(\langle u_s, v_s \rangle + \langle v_s, u_s \rangle)$$

and also that $2F$ and Φ are the momentum maps associated with the respective groups written in (6.34). Thus the isoenergetic orbit space is looked upon as the reduced phase space by the group $\mathbf{R} \times U(1)$. We denote the natural projection and the isoenergetic orbit space by π_μ^0 and M_μ^0, respectively:

$$\pi_\mu^0 : F^{-1}(4\kappa) \cap \Phi^{-1}(\mu) \longrightarrow M_\mu^0 := F^{-1}(4\kappa) \cap \Phi^{-1}(\mu)/\mathbf{R} \times U(1) . \tag{6.35}$$

Clearly, the manifold $F^{-1}(4\kappa) \cap \Phi^{-1}(\mu)$ is determined by the conditions $\langle u_s, u_s \rangle = 8\kappa$ and $\langle u_s, v_s \rangle + \langle v_s, u_s \rangle = 2\mu$. Hence the group G_S^0 given in (5.19) is found to act on it. We will show that this action is transitive. Take a fixed point (u_0, v_0) and an arbitrary point (u_s, u_s) of $F^{-1}(4\kappa) \cap \Phi^{-1}(\mu)$, where (u_0, v_0) is given by

$$u_0 = \begin{pmatrix} a \\ 0 \end{pmatrix} \qquad v_0 = \begin{pmatrix} b \\ 0 \end{pmatrix} \qquad \text{with} \quad a = 2\sqrt{2\kappa},\ b = \frac{\mu}{2\sqrt{2\kappa}} .$$

We here define two vectors u' and v' to be

$$u' = \frac{2a - b}{2a^2} u_s + \frac{1}{2a} v_s \qquad v' = \frac{b}{2a^2} u_s - \frac{1}{2a} v_s .$$

We then choose 2×2 matrices A and B so that the first-column vectors of A and B may be u' and v', respectively, and further that A and B may satisfy, along with suitably chosen second-column vectors, the conditions in (5.19). Then one has

$$\begin{pmatrix} A + B & 0 \\ -2B & A + B \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} u_s \\ v_s \end{pmatrix} .$$

This proves the transitivity, as is wanted. Thus we obtain

$$F^{-1}(4\kappa) \cap \Phi^{-1}(\mu) = \{gw_0; g \in G_S^0\}$$

where $w_0 = \binom{u_0}{v_0}$. Applying the momentum map (6.32) to this set results in

$$K^0(F^{-1}(4\kappa) \cap \Phi^{-1}(\mu)) = \{\mathrm{Ad}_g K^0(w_0); g \in G_S^0\} . \tag{6.36}$$

This shows that the image of K^0 is a (co-)adjoint orbit of $G^0_S \cong SU(2) \ltimes \mathbf{R}^3$, which we denote by $\mathcal{O}^0$.

Writing ι^0_μ for the inclusion map: $F^{-1}(4\kappa) \cap \Phi^{-1}(\mu) \to \mathbf{C}^4$, we consider the composite map

$$K^0 \circ \iota^0_\mu : F^{-1}(4\kappa) \cap \Phi^{-1}(\mu) \longrightarrow \mathcal{O}^0 . \tag{6.37}$$

Suppose now that $K^0 \circ \iota^0_\mu(u_s, v_s) = K^0 \circ \iota^0_\mu(u'_s, v'_s)$. In other words, one has $u'_s u'^*_s = u_s u^*_s$ and $u'_s v'^*_s + v'_s u'^*_s = u_s v^*_s + v_s u^*_s$. Then we can show that (u'_s, v'_s) and (u_s, v_s) lie in the same orbit of $\mathbf{R} \times U(1)$, that is, they are related by the action of (6.34). Hence $K^0 \circ \iota^0_\mu$ induces a diffeomorphism

$$\widetilde{K}^0_\mu : F^{-1}(4\kappa) \cap \Phi^{-1}(\mu)/\mathbf{R} \times U(1) \longrightarrow \mathcal{O}^0 \tag{6.38}$$

which satisfies

$$\widetilde{K}^0_\mu \circ \pi^0_\mu = K^0 \circ \iota^0_\mu . \tag{6.39}$$

We are to show that $\widetilde{K}^0_\mu$ is symplectomorphic. To this end, we first have to provide $\mathcal{O}^0$ with the KKS form. In keeping with the altered inner product (6.31), the KKS form on $\mathcal{O}^0$ is taken, at $\nu \in \mathcal{O}^0$, to be

$$\begin{aligned}\omega^0(\xi_Q, \eta_Q)(\nu) &= -\frac{\mathrm{i}}{2}\mathrm{tr}(\nu G_s[\xi, \eta]) \\ &= \mathrm{i}\,\mathrm{tr}((\nu_1 + \nu_2)([\xi_2, \eta_1 + \eta_2] + [\xi_1 + \xi_2, \eta_2]) + \nu_2[\xi_1 + \xi_2, \eta_1 + \eta_2]) .\end{aligned} \tag{6.40}$$

Further, note that the symplectic form on $\mathbf{C}^4$ is expressed, from (6.28), as

$$\mathrm{d}\theta = \frac{\mathrm{i}}{2}\mathrm{tr}(\mathrm{d}v_s \wedge \mathrm{d}u^*_s + \mathrm{d}u_s \wedge \mathrm{d}v^*_s) . \tag{6.41}$$

By using (6.40) and (6.41) together with $\nu = K^0(w_s)$, we can prove that

$$(K^0 \circ \iota^0_\mu)^* \omega^0 = (\iota^0_\mu)^* \mathrm{d}\theta . \tag{6.42}$$

Equations (6.39) and (6.42) are used to yield

$$(\widetilde{K}^0_\mu)^* \omega^0 = \sigma^0_\mu \tag{6.43}$$

where σ^0_μ is the reduced symplectic form defined on the reduced phase space M^0_μ through $(\pi^0_\mu)^* \sigma^0_\mu = (\iota^0_\mu)^* \mathrm{d}\theta$.

Theorem 3. All the isoenergetic orbit spaces of the MIC–Kepler problem are (co-)adjoint orbits of the respective symmetry groups; the isoenergetic orbit space for negative, positive, and zero energies are realized as a (co-)adjoint orbit of $G^- \cong SU(2) \times SU(2)$, of $G^+ \cong SL(2, \mathbf{C})$, and of $G^0 \cong SU(2) \ltimes \mathbf{R}^3$, respectively. Further, as reduced phase spaces, those isoenergetic orbits spaces are symplectomorphic with respective (co-)adjoint orbits endowed with the KKS form.

7. Perturbed MIC–Kepler problems

We consider a class of perturbed MIC–Kepler problems, which are compatible with the group theoretical treatment. Let us be given a perturbed Hamiltonian on $T^*\dot{\mathbf{R}}^4$,

$$H_c^{(\varepsilon)} = H_c + \varepsilon N^{(\varepsilon)} \tag{7.1}$$

where H_c is the Hamiltonian of the conformal Kepler problem given in (5.1). We assume further that $N^{(\varepsilon)}$ is in normal form with H_c, $\{H_c, N^{(\varepsilon)}\} = 0$, and invariant under the $U(1)$ action. Then the Hamiltonian (7.1) is reduced to

$$H_\mu^{(\varepsilon)} = H_\mu + \varepsilon N_\mu^{(\varepsilon)} \tag{7.2}$$

where $H_\mu^{(\varepsilon)}$ and $N_\mu^{(\varepsilon)}$ are reduced functions determined through $H_\mu^{(\varepsilon)} \circ \pi_\mu = H_c^{(\varepsilon)} \circ \iota_\mu$ and $N_\mu^{(\varepsilon)} \circ \pi_\mu = N^{(\varepsilon)} \circ \iota_\mu$, respectively. Further, H_μ and $H_\mu^{(\varepsilon)}$ prove to be in normal form, $\{H_\mu, H_\mu^{(\varepsilon)}\} = 0$, as a consequence of $\{H_c, H_c^{(\varepsilon)}\} = 0$, where the Poisson brackets of the reduced functions are taken with respect to the reduced symplectic form σ_μ. Hence the Hamiltonian flow for $H_\mu^{(\varepsilon)}$ runs on the energy manifold $H_\mu^{-1}(E)$. Further, since the flow for $H_\mu^{(\varepsilon)}$ and for H_μ commute, the flow for $H_\mu^{(\varepsilon)}$ induces a flow on the isoenergetic orbit space $H_\mu^{-1}(E)/G_t$.

In the case of $E < 0$, the isoenergetic orbit space is a (co-)adjoint orbit of $G^- \cong SU(2) \times SU(2)$. Since a generic (co-)adjoint orbit of $SU(2) \times SU(2)$ is diffeomorphic with $S^2 \times S^2$, the perturbed Hamiltonian $H_\mu^{(\varepsilon)}$ defines a flow on $S^2 \times S^2$. The Euler number of $S^2 \times S^2$, equal to four, gives the number of singular points of the flow, which in turn is the number of closed orbits for the perturbed MIC–Kepler problem $H_\mu^{(\varepsilon)}$ with negative energy. For the perturbed Kepler problem, this result is already known and has been given by Moser (1970).

We now consider the perturbed Hamiltonian induced on the isoenergetic orbit space. Let us be reminded that the Hamiltonian H_c can be switched to A_λ when the energy manifold $H_c^{-1}(E)$ for $E < 0$ is taken into account (see (5.6)). Further, note that $4rX_{H_c} = X_{A_\lambda}$ on $H_c^{-1}(E)$. Therefore, a perturbed Hamiltonian $H_c^{(\varepsilon)}$ which is in normal form with H_c and hence invariant under the flow $\exp(tX_{H_c})$ should be invariant under the flow of $\exp(tX_{A_\lambda})$, the action of (5.12). Thus $H_c^{(\varepsilon)}$ must be invariant under the action of the product group $U(1) \times U(1)$ of (3.7) and (5.12), when restricted on $H_c^{-1}(E)$ and therefore on $H_c^{-1}(E) \cap \Phi^{-1}(\mu)$. The $U(1) \times U(1)$ invariance makes $H_c^{(\varepsilon)} | H_c^{-1}(E) \cap \Phi^{-1}(\mu)$ project to a function on the isoenergetic orbit space M_μ^- (see (6.5)), which function we denote by $H_\mu^{(\varepsilon)-}$. Incidentally, since M_μ^- is diffeomorphic with a (co-)adjoint orbit in $\mathcal{G}^- \cong su(2) \oplus su(2)$, $H_\mu^{(\varepsilon)-}$ must be a function of $K^-(w)$ (see (6.8)), so that it becomes a function of $\mathrm{tr}(K^-(w)e_m)$ with e_m a basis of $\mathcal{G}^-$. Thus $H_\mu^{(\varepsilon)-}$ is a function of $\mathrm{tr}(K_{\mathrm{L}}^-(u)\sigma_j)$ and $\mathrm{tr}(K_{\mathrm{R}}^-(v)\sigma_j)$, where σ_j are the Pauli matrices. Hence it can be expressed in terms of J_{j+1} and J_{j+4}, $j = 1, 2, 3$, given in (3.20).

In the positive- and zero-energy cases, perturbed Hamiltonians are also obtained in the same manner. For $E > 0$, the perturbed Hamiltonian $H_\mu^{(\varepsilon)+}$ on the isoenergetic orbit space M_μ^+ must be a function of $K^+(w)$, and hence of $\mathrm{tr}(U\sigma_j)$ and $\mathrm{tr}(V\sigma_j)$ (see (6.15)), so that it can be expressed in terms of $J_{j+1} + J_{j+4}$ and J_{j+8}, $j = 1, 2, 3$.

For $E = 0$, the perturbed Hamiltonian $H_\mu^{(\varepsilon)0}$ on the isoenergetic orbit space M_μ^0 should be a function of $K^0(w_s)$, and hence of $\mathrm{tr}(U_s\sigma_j)$ and $\mathrm{tr}(W_s\sigma_j)$ (see (6.30)), so that it can be written in terms of $J_{j+1}+J_{j+4}$ and $J_{j+1}-J_{j+4}-2J_{j+8}$, $j=1,2,3$.

8. Concluding remarks

In this section, we make remarks on the co-adjoint structures studied in sections 4 and 6. Each of those co-adjoint structures is a specialization of the following theorem in part.

Theorem 4. Let (M,ω) be a Hamiltonian $G = H\times K$-space, where ω is a symplectic form on M, and H and K are connected Lie groups. Let $\phi_H\oplus\phi_K: \ M\rightarrow\mathcal{H}^*\oplus\mathcal{K}^*$ be the equivariant momentum maps associated with the $H\times K$ action, where $\mathcal{H}$ and $\mathcal{K}$ are Lie algebras of H and K, respectively, and $\mathcal{H}^*$ and $\mathcal{K}^*$ are their duals. Assume that $\mu\in\mathcal{K}^*$ is a regular value of ϕ_K, and the reduced phase space $\phi_K^{-1}(\mu)/K_\mu$ is a manifold, where K_μ is the isotropy subgroup of K at μ. Assume further that $H\times K_\mu$ acts transitively on $\phi_K^{-1}(\mu)$. Then $\phi_H(\phi_K^{-1}(\mu))$ is a co-adjoint orbit $\mathcal{O}$ of H, and $\phi_H:\phi_K^{-1}(\mu)\rightarrow\mathcal{O}$ projects to the quotient to define a symplectic covering map $\widetilde{\phi}_H:\phi_K^{-1}(\mu)/K_\mu\rightarrow\mathcal{O}$, where $\mathcal{O}$ is considered as a symplectic manifold equipped with the Kirillov–Kostant–Souriau form.

This theorem is stated in Kummer (1983). See also Iwai (1990) for a brief proof. In the latter paper, another example of interest is described.

In section 4, we have only to apply this theorem with $K = U(1)$ and $H = SU(2,2)$. It then follows that $\widetilde{K}_\mu$ given by (4.4) is a symplectic covering map. We have already shown that $\widetilde{K}_\mu$ is a symplectomorphism.

In section 6, theorem 4 should be applied for pairs $K = U(1)\times U(1)$ and $H = G^-\cong SU(2)\times SU(2)$, $K=\mathbf{R}\times U(1)$ and $H = G^+\cong SL(2,\mathbf{C})$, and $K=\mathbf{R}\times U(1)$ and $H=G^0\cong SU(2)\ltimes\mathbf{R}^3$. Our results in section 6 are that the symplectic covering map stated in the theorem is indeed a symplectomorphism for the respective pairs of H and K.

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